A New Solution to the Inverse Problem of Optimal Regulator Control

In chemical process control, the application of an optimal control strategy, dictated by the theory of the linear quadratic problem, does not always guarantee the best process performance. The main problem is that the unspecified weighting matrices in the quadratic performance index have a pronounced effect on the optimal control law derived from linear quadratic theory. In the past, values of the weighting matrices have been selected by experimenting with the process in much the same manner as a PID controller is tuned. The objective of the present study is to resolve the inverse problem of optimal regulator control, limiting the performance index in the quadratic form with two independent weighting matrices which are investigated. A new criterion has been developed such that the computation of determining those matrices can be carried out off-line in advance, and then stored in a computer for real-time use.

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Introduction

The solutions to the optimal control problem, based on such theories as dynamic programming (Sage and White, 1977) or the minimum (maximum) principle (Ray, 1981), were well developed by many investigators during the 1960's.

In practice, however, the performance index has some uncertainties that exist in its form and unspecified parameters. Since those uncertainties strikingly influence the design of a control system, the optimal control law, dictated by the preceding techniques, does not always give the best process performance. Thus, the existence of the uncertainties in the performance index leads to an investigation of the inverse problem of optimal control, and its solution removes the uncertainties in the application of optimal control theory.

Broadly speaking, the inverse problem of optimal control theory may be stated as follows: Given a completely controllable model of a plant and the control law, determine all performance indices (if any) for which the control law is optimal (Kalman, 1964). In the light of this definition, it seems very difficult to find the general solution that follows a given control law. For simplicity, the inverse problem is sometimes narrowed to a special case: The form of the performance index has been specified, with only parameters investigated (Denn, 1967).

Kalman (1964) first studied an inverse problem for the linear quadratic system with a scalar control variable. He discussed the relationship between optimality and stability, and developed a criterion to determine the optimal weighting matrix in the quadratic performance index for a given linear control law.

Denn (1967, 1969) investigated the inverse problem in a general linear system, and even in a special nonlinear system with a linear switching control law. By integrating the canonical equations, he found that the performance index in this case is just of quadratic type.

Smith (1972) gave an example in the control of a distillation column to illustrate how an optimal controller may be designed for disturbance changes. He merely mentioned that the value of γ in performance index is essentially a tuning parameter determined by experimenting with the process, in much the same manner as PID controllers are tuned.

Sage and White (1977) mentioned that the second variation and neighboring optimal methods of control law computation provide a method for choosing the proper form of weighting matrices in the performance index. This method, however, does not specify the values of the matrices; in particular, it does not work for a linear system.

Ray (1981) discussed optimal servo control for a continuous stirred-tank reactor (CSTR). He pointed out that the effects of optimal servo control are also strongly related to the value of the parameter in the performance index.

In this paper, the inverse problem is studied for a linear regulator control system with an integral and a proportional-integral control law. The relationship between optimality and stability is

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analyzed. Based on this, a new approach has been developed to determine the optimal weighting matrices in the quadratic performance index. In application for a given system and specified control law, the weighting matrices can be exactly determined *a priori* without the need of upsetting the system by tuning experiments.

Problem Statement

The optimal regulator control of a time-invariant linear system (TILS) can be stated as follows. In the class of piecewise continuous functions, find a vector control $u(t)$ that minimizes the quadratic performance index:

$$J = \frac{1}{2}\int_{t_o}^{t_e} (\Delta x^T F \Delta x + \Delta u^T E \Delta u)dt \tag{1a}$$

subject to the constraints

$$\dot{x} = Ax + Bu + \Gamma d \tag{1b}$$

$$x(t_o) = x_o = x_e = x(t_e) \tag{1c}$$

$$u(t) \epsilon U^m,\ t \epsilon [t_o, t_e] \tag{1d}$$

where: x is an n state vector; u is an m control vector, $m \leq n$; $\Delta x = x - x_e$; $\Delta u = u - u_e$; superscript T denotes transpose; d is a p disturbance vector; F is an $n \times n$ nonnegative definite symmetric matrix; E is an $m \times m$ positive definite symmetric matrix; A, B, and Γ are $n \times n$, $n \times m$, and $n \times p$ matrices, respectively; and U^m is a convex subset of m-dimensional Euclidean space.

Introduce an auxiliary state vector containing the disturbances:

$$y = u + Dd \tag{2a}$$

where

$$D = B^+ \Gamma \tag{2b}$$

B^+ is called the pseudoinverse of matrix B, and is defined as

$$B^+ = (B^T B)^{-1} B^T \quad (\text{Rank B} = m \leq n) \tag{2c}$$

The system of Eq. 1 is then converted into an equivalent system (Johnson, 1968; Ray, 1981):

$$J_r = \frac{1}{2}\int_{t_o}^{t_e} (\Delta z^T F_r \Delta z + \Delta v^T E_r \Delta v)dt \tag{3a}$$

subject to the constraints

$$\dot{z} = Pz + Qv \tag{3b}$$

$$z = \begin{bmatrix} x \\ y \end{bmatrix};\quad z_o = \begin{bmatrix} x_o \\ y_o \end{bmatrix};\quad z_e = \begin{bmatrix} x_o \\ y_e \end{bmatrix} \tag{3c}$$

$$v \triangleq \dot{y} = \dot{u} + D\dot{d} \tag{3d}$$

where: $\Delta z = z - z_e$; $\Delta v = v - v_e$; $y_o = u_o + Dd_o$; $y_e = u_e + Dd_e$; F_r is an $(n + m) \times (n + m)$ nonnegative definite symmetric matrix; E_r is an $m \times m$ positive definite symmetric matrix; and

$$P = \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right],\quad (n + m) \times (n + m) \text{ matrix} \tag{3e}$$

$$Q = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right],\quad (n + m) \times m \text{ matrix} \tag{3f}$$

(I is the unit matrix).

The control law, corresponding to the system of Eq. 1, or equivalently to the system of Eq. 3, is given by the integral scheme:

$$\Delta u = \int_{t_o}^{t} \Pi_1 \Delta x(\tau) d\tau + \Pi_2 \tag{4a}$$

where

$$\Pi_1 = -\Phi(t, \tau) K_1(\tau) \tag{4b}$$

$$\Pi_2 = \Phi(t, t_o)\Delta y_o - D\Delta d \tag{4c}$$

$$(\Delta y_o = y_o - y_e; \quad \Delta d = d - d_e)$$

$\Phi(t, t_o)$ is an $m \times m$ matrix, defined by:

$$\dot{\Phi}(t, t_o) = -K_2(t)\Phi(t, t_o); \quad \Phi(t_o, t_o) = \text{I} \tag{4d}$$

where $K_2(t)$ is an $m \times m$ matrix and $K_1(t)$, an $m \times n$ matrix. They are elements of the feedback control gain $K(t)$ in the system of Eq. 3:

$$K(t) = [K_1(t)\ K_2(t)],\quad m \times (n + m) \text{ matrix} \tag{4e}$$

and

$$\Delta v = -K(t)\Delta z \tag{4f}$$

Note that if all disturbances in a TILS are step changes, then $\Delta d = \Delta d_o = 0$, and $\Pi_2 = \Phi(t, t_o)\Delta u_o$. This implies that the regulator control law, Eq. 4, does not depend on the intensity (or magnitude) of the step changes in disturbances, as Johnson (1968) indicated.

With this preliminary analysis, the inverse problem in the paper can be stated as: Given a completely controllable system, Eq. 1, or equivalently Eq. 3, and the control law in Eq. 4, determine all matrices F_r and E_r in the quadratic performance index, Eq. 3a, such that the control law is optimal.

Stability and Offset of a Regulator TILS

The following assumptions are made:

1. The system studied here is completely controllable.
2. All inputs and outputs of the system are bounded (BIBO), i.e.,

$$\|x(t)\|,\ \|u(t)\|, \text{ and } \|d(t)\| < \infty, \quad \forall t \epsilon [t_o, t_e]$$

Note that $x(t)$, $u(t)$, and $d(t)$ may have a finite number of discontinuous points in the time domain, $t \epsilon [t_o, t_e]$.

3. The equilibrium time t_e is sufficiently large to ensure that $K(t)$ is a constant (Ray, 1981).

First, from assumption 2, the Laplace transform of Δx exists in the s domain, $0 < s < \infty$.

Second, since the system is BIBO, i.e., for a sufficiently large number $M > 0$:

$$\|\Delta \dot{x}\| = \|A\Delta x + B\Delta u + \Gamma \Delta d\| \leq M, \quad \forall t \epsilon [0, \infty)$$

Thus, for any $\epsilon > 0$ and $s \geq \epsilon$,

$$\|\Delta \dot{x}\, e^{-st}\| = \|\Delta \dot{x}\| e^{-st} \leq M e^{-\epsilon t}, \quad \forall t \epsilon [0, \infty)$$

This implies that the Laplace transform of $\Delta \dot{x}$

$$\int_o^\infty \Delta \dot{x} e^{-st} dt = \int_o^\infty (A\Delta x + B\Delta u + \Gamma \Delta d) e^{-st} dt$$

is uniformly convergent for all $s > 0$. Then the final-value theorem follows immediately:

$$\lim_{t \to \infty} \Delta x(t) = \lim_{s \to 0} s\Delta x(s) \tag{5}$$

Note that it the TILS is not BIBO, the final-value theorem may not be true.

Now suppose that the control law, Eq. 4, has been optimized. By assumption 2 above, the system is BIBO under the optimal integral regulator control, Eq. 4. Taking the Laplace transform of both sides of Eq. 1b gives

$$\begin{gathered} s\Delta x(s) = A\Delta x(s) + B\Delta u(s) + \Gamma \Delta d(s) \\ (\text{Note that } \Delta x_o = x_o - x_e = 0) \end{gathered} \tag{6}$$

Also, taking the Laplace transform of both sides of Eq. 4a and using the convolution theorem [by assumption 3, $K(t)$ is the steady state gain of the controller] one obtains:

$$\begin{aligned} \Delta u(s) = &-(sI + K_2)^{-1} K_1 \Delta x(s) \\ &+ (sI + K_2)^{-1} \Delta y_o - D\Delta d(s). \end{aligned} \tag{7}$$

Substituting Eq. 7 into Eq. 6 and rearranging results in:

$$\begin{aligned} &s \cdot \Delta x(s) = \\ &s \cdot [sI - A + B(sI + K_2)^{-1} K_1]^{-1} \\ &\cdot B(sI + K_2)^{-1} \Delta y_o + [sI - A \\ &+ B(sI + K_2)^{-1} K_1]^{-1} (\Gamma - BD) \cdot s \cdot \Delta d(s). \end{aligned} \tag{8}$$

Assuming K_2 and $BK_2^{-1}K_1 - A$ to be nonsingular, it directly follows that

$$\lim_{s \to 0} [sI - A + B(sI + K_2)^{-1} K_1]^{-1} = (BK_2^{-1} K_1 - A)^{-1} \tag{9a}$$

$$\lim_{s \to 0} (sI + K_2)^{-1} = K_2^{-1} \tag{9b}$$

$$\lim_{s \to 0} s \cdot \Delta d(s) = \alpha \nrightarrow \pm \infty. \tag{9c}$$

If Δx_∞ designates the offset of the system, $\Delta x_\infty = \lim_{t \to \infty} \Delta x(t)$, then combining Eqs. 8 and 9 leads to the final result:

$$\Delta x_\infty = (BK_2^{-1} K_1 - A)^{-1} (\Gamma - BD)\alpha. \tag{10}$$

Equation 10 shows that if a TILS satisfies all three assumptions, and K_2 and $BK_2^{-1}K_1 - A$ are nonsingular, then the optimal integral control law is stable.

Further, it can be seen from Eq. 10 that there are three factors that have an effect on the offset of the system:

1. If $\Gamma = BD$, i.e., $\dim(B) = n$, then $\Delta x_\infty = 0$, or the system is stable with no offset.

2. If $\Gamma \neq BD$, and $\alpha = 0$, then the system is also stable with no offset. For instance, when disturbances are decaying signals such as $e^{-\lambda t} \cos \omega t (\lambda, \omega > 0)$, it is easily shown in this case that $\alpha = 0$. Therefore, a system that is disturbed by those kinds of signals is always stable, with no offset.

3. If $\Gamma \neq BD$, and $\alpha \neq 0$, then the system is stable, but with the offset Δx_∞ indicated by Eq. 10.

The remaining problem is the effect of $(BK_2^{-1}K_1 - A)^{-1}$ on Δx_∞. In terms of the linear-quadratic control theory, the weighting matrices F_r and E_r have a large influence on the controller gain $K = [K_1\ K_2]$. It is shown in practice that transient responses to disturbances are very sensitive to changes in the values of F_r and E_r (see the Application section below). In other words, it will take a long period of time to reach the equilibrium state if the values of F_r and E_r are not selected properly. Thus, the choice of the values of F_r and E_r is really the key issue in all optimal feedback control problems with a quadratic performance index.

Criterion for Evaluation of F_r and E_r

Since the steady state controller gain $K = [K_1\ K_2]$ depends on F_r and E_r in the quadratic performance index, Eq. 3a, for simplicity, define an $n \times n$ nonsingular matrix:

$$H = BK_2^{-1} K_1 - A. \tag{11}$$

then H also depends on F_r and E_r. Again define an n vector:

$$\beta = (\Gamma - BD)\alpha \tag{12}$$

that is entirely independent of F_r and E_r. Thus, Eq. 10 can be simplified to a determined linear equation:

$$H\Delta x_\infty = \beta. \tag{13}$$

Taking the variation in both sides of Eq. 13 gives

$$H \cdot \delta \Delta x_\infty + \delta H \cdot \Delta x_\infty = \delta \beta$$

or

$$\delta \Delta x_\infty = H^{-1} \delta \beta - H^{-1} \cdot \delta H \cdot \Delta x_\infty.$$

Now perform the evaluation of norms in the above equation:

$$\begin{aligned} \|\delta \Delta x_\infty\| &\leq \|H^{-1} \delta \beta\| + \|H^{-1} \cdot \delta H \cdot \Delta x_\infty\| \\ &\leq \|H^{-1}\| \|\delta \beta\| + \|H^{-1}\| \|\delta H\| \|\Delta x_\infty\| \end{aligned}$$

or

$$\frac{\|\delta\Delta x_\infty\|}{\|\Delta x_\infty\|} \le \|H^{-1}\| \frac{\|\delta\beta\|}{\|\Delta x_\infty\|} + \|H^{-1}\| \, \|\delta H\|. \tag{14}$$

Note that from Eq. 13,

$$\|\Delta x_\infty\| \ge \frac{\|\beta\|}{\|H\|}. \tag{15}$$

Substitution of this inequality, Eq. 15, into Eq. 14 gives

$$\frac{\|\delta\Delta x_\infty\|}{\|\Delta x_\infty\|} \le \|H\| \, \|H^{-1}\| \frac{\|\delta\beta\|}{\|\beta\|} + \|H^{-1}\| \, \|\delta H\|$$

$$= \|H\| \, \|H^{-1}\| \frac{\|\delta\beta\|}{\|\beta\|} + \|H\| \, \|H^{-1}\| \frac{\|\delta H\|}{\|H\|}$$

or

$$\frac{\|\delta\Delta x_\infty\|}{\|\Delta x_\infty\|} \le \|H\| \, \|H^{-1}\| \left(\frac{\|\delta\beta\|}{\|\beta\|} + \frac{\|\delta H\|}{\|H\|} \right). \tag{16}$$

Equation 16 implies that the relative error of offset Δx_∞ is determined by two terms. The first is the relative error, $\|\Delta\beta\|/\|\beta\|$, caused by disturbances; the second is the relative error, $\|\delta H\|/\|H\|$, caused by F_r and E_r. However, the number

$$v = \|H\| \, \|H^{-1}\| \tag{17}$$

just plays the role of an "amplifier" in Eq. 16 to amplify the offset of the system. It can be proved mathematically (see Appendix) that

$$\nu = \|H\| \, \|H^{-1}\| = \frac{\max_i \sqrt{\lambda_i}}{\min_i \sqrt{\lambda_i}} \ge 1 \tag{18}$$

where $\lambda_i > 0$ stands for all eigenvalues of H^TH (since H is nonsingular, H^TH must be symmetric and positive definite, and all $\lambda_i > 0$).

In terms of Eq. 18, it now can be seen that if matrices F_r and E_r are not selected properly, the value of ν could be much greater than 1. In this case, there either would be a large offset (if any), or a long period of time would be required to bring the system to the equilibrium state. It can be seen from Eqs. 16 and 18 that theoretically the best value of v is unity; however, for a physical system this may not be the case. Therefore, to obtain the optimal regulator control of a given system, F_r and E_r must be determined such that v attains its minimum value:

$$\nu_{\min} = \min_{\substack{F_r \in R^{n+m} \\ E_r \in R^m}} \frac{\max_i \sqrt{\lambda_i}}{\min_i \sqrt{\lambda_i}} (F_r, E_r) \ge 1. \tag{19}$$

Obviously, this condition gives the minimum offset (if any). On the other hand, in terms of Eq. 8, the matrix, $H(s)$,

$$H(s) = sI - A + B(sI + K_2)^{-1}K_1$$

dominates the performance of a dynamic process. More precisely, we may state that if

$$\|H(s)\| \, \|H^{-1}(s)\| \, (F_r, E_r) = \min \text{ for all } s > 0$$

then the system possesses a robust ability to buffer all error disturbances, and has the best dynamic response. Since $H(s)$ approaches H as $s \to 0$, the above condition may be rewritten as:

$$\frac{\max_i \sqrt{\lambda_i} + \epsilon_1(s)}{\min_i \sqrt{\lambda_i} + \epsilon_2(s)} (F_r, E_r) = \min \text{ for all } s > 0$$

where $\epsilon_1(s)$ and $\epsilon_2(s) \to 0$ as $s \to 0$. This implies that the condition in Eq. 19 would also always guarantee satisfactory dynamic performance. Note that F_r and E_r are subject to R^{n+m} and R^m, respectively. Therefore, $\nu_{\min}$ may be a relative minimum, and all relative minima of $\nu_{\min}$'s theoretically give the complete solution set of F_r and E_r in R^{n+m} and R^m.

H in the Case of the Optimal Proportional-Integral Regulator Control

Instead of the integral control law, Eq. 4, the control law, corresponding to the system of Eq. 1 or Eq. 3, now takes the form of a proportional-integral scheme:

$$\Delta u = \Omega_1 \Delta x + \int_{t_o}^{t} \Omega_2 \Delta x dt + \Omega_3 \tag{20a}$$

where

$$\Omega_1 = -K_2 B^+ \tag{20b}$$

$$\Omega_2 = K_2 B^+ A - K_1 \tag{20c}$$

$$\Omega_3 = \Delta y_o - D\Delta d. \tag{20d}$$

Likewise, the analysis in the frequency domain shows that:

$$s\Delta x(s) = s(s^2 I - sA - sB\Omega_1 - B\Omega_2)^{-1} \cdot [B\Delta y_o + (\Gamma - BD)s\Delta d(s)]. \tag{21}$$

If $\dim(B) = n$, Eq. 21 is equivalent to Eq. 8. In this particular case, it implies that the state responses under the optimal proportional-integral regulator control are exactly the same as that under optimal integral regulator control (both have the same steady state gain).

Since the system is BIBO, $\mathrm{Lim}_{s\to 0}\, s\Delta d(s) = \alpha \nrightarrow \pm\infty$. Further, assuming $B\Omega_2$ to be nonsingular, then

$$\lim_{s\to 0} (s^2 I - sA - sB\Omega_1 - B\Omega_2)^{-1} = -(B\Omega_2)^{-1} \tag{22}$$

Therefore

$$\Delta x_\infty = 0 \tag{23}$$

This means that the optimal proportional-integral control law is always stable with no offset if a TILS satisfies the three original assumptions and the assumption that $B\Omega_2$ is nonsingular. However, it must be pointed out that the assumption of $B\Omega_2$ being nonsingular is very important. Otherwise—for instance in a scalar control system (that is, u is a scalar) in which the single-column matrix, B, takes the form $B = [0\ 1]^T$—$B\Omega_2$ is singular and the system is stable with the offset indicated by Eq. 21 as $s \rightarrow 0$. Therefore, optimal proportional-integral control has very little advantage over optimal integral control except that $B\Omega_2$ is nonsingular.

According to Eqs. 21 and 22, the criterion of Eq. 19 is still available in the case of the optimal proportional-integral regulator control; however, the matrix H, which dominates the state transient responses and system stability, is given by:

$$H = B\Omega_2 = B(K_2B^+A - K_1). \tag{24}$$

Note that although H as defined by Eq. 24 is slightly different from that defined by Eq. 11, both of these equations play the same role in determining the optimal weighting matrices by using the criterion of Eq. 19.

Application

Now consider a continuous stirred-tank reactor (CSTR) in which an isothermal multicomponent chemical reaction is being carried out. A sketch of the system is shown in Figure 1. Note that component A enters the reactor in two streams, one of which is assumed to be the source of the disturbance. The reaction system is described by:

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B \xrightarrow{k_3} C$$

$$A \xrightarrow{k_4} C$$

Defining the concentrations c_A and c_B as state variables, concentrations c_{Af} and c_{Bf} as control variables, and concentration c_{Ad} as the disturbance variable, a two-dimensional TILS was formed. Further, by introducing a new state vector, Eq. 2a, an equivalent four-dimensional TILS can be reformed, as expressed by Eq. 3.

The effects of F_r and E_r were studied in a series of computer simulation tests. Three different diagonal matrices for F_r were chosen to form three base cases; for each of these cases, a series of diagonal matrices, E_r, was selected to determine the effects of various F_r–E_r pairs on ν, the criterion of Eq. 19. In all cases where ν increased, the system responses became increasingly poor, and eventually unstable. On the other hand, the responses were very satisfactory as long as ν reached its minimum value (mostly, close to unity), as predicted by the theory. In particular, the control action required decreased and the control became tighter as F_r increased.

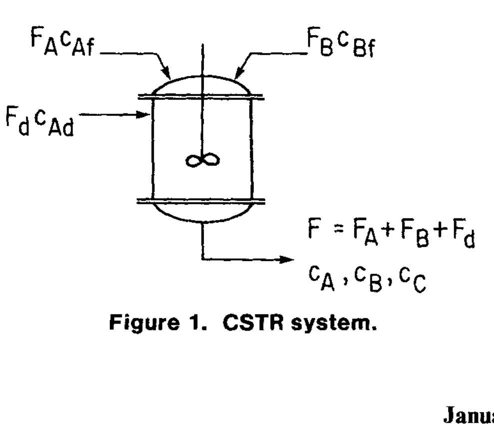

Figure 1. CSTR system.

It was further shown that all optimal pairs of F_r and E_r can be calculated in advance by using the criterion of Eq. 19, without bringing the system to an unstable state by tuning experiments, because they are merely determined by the eigenvalues of matrix H. Once the controller gain K has been found by solving the Riccati equation, it can be stored in the computer for real-time use.

Finally, in many industrially important processes the parameters of the system are not known precisely, and are known only to lie within a particular interval. However, it is noted that the relative error, $\|\delta H\|/\|H\|$, represents the stochastic uncertainty affected not only by F_r and E_r (extrinsic selection), but also by A and B (intrinsic parameter matrices of the system). In connection with Eqs. 16 and 18, therefore, it turns out that the criterion of Eq. 19 would guarantee the best system responses under disturbances of any kind of errors, intrinsic and extrinsic.

Conclusions

Based on the stability analysis of TILS, a criterion for determining the unspecified matrices in the quadratic performance index has been developed. A large number of computer simulation results demonstrate that the criterion provides a good approach for evaluating F_r and E_r. These values provide excellent responses when a system under optimal integral control undergoes a step change in disturbance. The stability analysis also indicates that the optimal values of F_r and E_r are entirely independent of external disturbances, and they are essentially determined by the intrinsic properties of the system. Therefore, they can be calculated off-line in advance by the criterion of Eq. 19 without the need for upsetting the system by tuning experiments.

Notation

A = $n \times n$ matrix, or reactant
B = $n \times m$ matrix, or reactant
c = concentration
C = reactant
d = p disturbance vector
D = $m \times p$ matrix
E = $m \times m$ weighting matrix
F = $n \times n$ weighting matrix
H = $n \times n$ nonsingular matrix, Eqs. 11, 24
I = unit matrix
J = quadratic functional
k_i = reaction rate constants
K = $m \times (n + m)$ steady state gain matrix of controller, and $K = [K_1K_2]$
K_1 = $m \times n$ matrix
K_2 = $m \times m$ matrix
m, n, p = dimension of u, x, d
P = $(n + m) \times (n + m)$ matrix
Q = $(n + m) \times m$ matrix
R^{n+m}, R^m = $(n + m) \times (n + m)$ and $m \times m$ matrix sets
s = parameter in Laplace transform
t = time
u = m control vector
U^m = convex subset of m-dimensional Euclidean space
v = m reformed control vector
x = n state vector
y = m reformed state vector
z = $n + m$ reformed state vector

Greek letters

α = steady state of disturbance
β = vector, Eq. 12
Γ = $n \times p$ matrix
γ = eigenvalue of matrix
ν = scalar, Eq. 17
Φ = state transition matrix, Eq. 4d

Subscripts and superscripts

A, B, C = component of reactants
d = disturbance
e = equilibrium condition
f = feed component
i = component i
r = reform
T = transpose of matrix
o = initial state
$+$ = pseudoinverse of matrix
∞ = infinite time

Appendix

Much of the theory of this paper utilizes the concept of a normed linear space, which is discussed by Porter (1967).

Let A be an $m \times n$ linear transformation matrix, and Rank A = n, ($m \geq n$). If a real number A is defined as

$$\|A\| = \max_{\|x\|=1} \|Ax\| \tag{A1}$$

where

$$\|Ax\| = \left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2\right]^{1/2} \tag{A2}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then it can be proved that the real number $\|A\|$ expressed by Eq. A1 is the norm of linear transformation matrix A.

Theorem 1.

If $\|A\|$ is the norm of an $m \times n$ matrix A, and Rank A = $n(m \geq n)$, then

$$\|A\| = \max_i \sqrt{\lambda_i} \quad \lambda_i > 0 \tag{A3}$$

where λ_i ($i = 1, 2, \ldots, n$) are the eigenvalues of matrix A^TA.

Substituting Eq. A2 into Eq. A1 gives

$$\|A\| = \max\left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2\right]^{1/2}.$$

subject to constraint $\|x\| = 1$. Note that

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2 = (Ax)^TAx = x^TA^TAx.$$

Therefore

$$\|A\|^2 = \max x^TA^TAx \tag{A4}$$

subject to the constraint $\|x\|^2 = x^Tx = 1$.

To solve this constrained optimization problem, the method of LaGrange multipliers is used; thus, the stationary point of Eq. A4 should satisfy the following stationary equation:

$$\nabla[x^TA^TAx + \lambda(1 - x^Tx)] = 0$$

or

$$2A^TAx - 2\lambda x = 0$$

or

$$A^TAx = \lambda x \tag{A5}$$

Equation A5 indicates that the stationary point of Eq. A4, x, is just the eigenvector of matrix A^TA, and the LaGrange multiplier, λ, just the eigenvalue of matrix A^TA.

Substituting Eq. A5 into Eq. A4 yields

$$\|A\|^2 = \max_{\|x\|=1} x^T\lambda x = \max_{\|x\|=1} \lambda x^Tx = \max \lambda$$

since Rank A = $n(m \geq n)$, A^TA is nonsingular and positive definite; that is, $x^TA^TAx > 0$. This means that all eigenvalues of A^TA are positive, $\lambda_i > 0$. Therefore,

$$\|A\| = \max_i \sqrt{\lambda_i} \quad (i = 1, 2, \ldots, n).$$

Theorem 1 follows immediately.

Theorem 2.

If $\|A\|$ is the norm of a nonsingular matrix A, then

$$\|A^{-1}\| = \frac{1}{\min_i \sqrt{\lambda_i}} \quad \lambda_i > 0 \tag{A6}$$

where λ_i ($i = 1, 2, \ldots, n$) are the eigenvalues of matrix A^TA.

According to theorem 1, $\|A^{-1}\| = \max_i \sqrt{\nu_i}$, where $\nu_i > 0$ ($i = 1, 2, \ldots, n$) are the eigenvalues of $(A^{-1})^TA^{-1}$. Note that,

$$(A^{-1})^TA^{-1} = (A^T)^{-1}A^{-1} = (AA^T)^{-1} = [(A^TA)^T]^{-1} = (A^TA)^{-1}.$$

Let λ_i be the eigenvalue of A^TA ($\lambda_i > 0$); then $\nu_i = (1/\lambda_i)$, that is,

$$\max_i \sqrt{\nu_i} = \frac{1}{\min_i \sqrt{\lambda_i}}.$$

Theorem 2 follows.

In terms of theorem 1 and theorem 2, the following formulas can be obtained immediately:

$$\|A\| \, \|A^{-1}\| = \frac{\max_i \sqrt{\lambda_i}}{\min_i \sqrt{\lambda_i}} \qquad \text{(A7)}$$

$$\|A^T A\| \, \|(A^T A)^{-1}\| = \frac{\max_i \lambda_i}{\min_i \lambda_i} \qquad \text{(A8)}$$

where $\lambda_i > 0$ ($i = 1, 2, \ldots, n$) are the eigenvalues of matrix $A^T A$, and A is an $n \times n$ nonsingular matrix.

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Manuscript received Nov. 2, 1984, and revision received June 5, 1986.